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THE CALCULATION OF SCATTERING BY THIN SHELLS

by

JOHN GREENSTADT

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John Greenstadt

Morris Kline

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Project Director

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ABSTRACT

This paper investigates the problem of calculating the electromagnetic field scattered by a thin shell whose surface is a quadric surface. The paper is an application of a general formula developed by J. B. Keller in Report No. 172-5, "Reflection and Transmission of Electromagnetic Waves by Thin Curved Shells". Parts of the paper are expository, dealing with differential geometry of surfaces, and in particular of quadric surfaces. The paper attempts no actual numerical computations but shows how these would be carried out in any numerical case. In one specialized case of some interest, the calculations can be carried through with literal coefficients, and this is done (see 5.9) p. 19). A final section of the paper considers the problem of finding the points on the surface which are responsible for the preponderant part of the field reflected to the observation point; these are the so-called specular points.

1. Introduction.

In his paper "Reflection and Transmission of Electromagnetic Waves by Thin Curved Shells", Research Report No. 172-5 to the Watson Laboratories of the United States Air Force, J.B. Keller derives certain formulas which give considerable insight into the nature of the radiation field of an electromagnetic source when an obstacle of somewhat restricted geometrical character is present in the otherwise homogeneous medium. To within the order of approximation shown in the paper, the formulas give the total electric and magnetic fields as explicit functions of certain surface and space parameters, and these field quantities are at least theoretically calculable. Owing to the inevitable complexity of the formulas, calculations based upon them are of necessity somewhat tedious.

The paper makes no contribution to the underlying theory nor does it attempt any simplification of the theory in Keller's paper. It does set forth the background material necessary for the calculation of reflected radiation fields by the use of Keller's formula and shows in some detail what the essential steps are like in the case of an ellipsoidal surface. One case of general interest, namely the case where source and observation point coincide (as in radar devices) and lie on one of the principal axes of the ellipsoid, can be carried through with literal coefficients.

The problem of arbitrary electromagnetic radiation incident on a simply connected homogeneous obstacle imbedded in a homogeneous medium is treated exactly by Keller, and a set of intractable linear integro-differential equations is written down whose implicit solution constitutes the exact radiation field. Keller has managed to obtain an approximate explicit solution for the case of an obstacle which is a thin shell of arbitrary shape, acting upon incident radiation from a source whose distance from the shell is large compared to the wavelength of the radiation. Keller's explicit formula, based on the so-called stationary phase approximation, gives the electric and magnetic field vectors at a point relatively far from the shell. These vectors are exhibited as the sums of two terms; one the incident field, and the other the field reflected or transmitted to the point in question from a given neighborhood on the shell. The formula, slightly rearranged, is given at the end of this introduction. The form of the Keller's solution for the electric field is as follows:

1.1) $$E = E_o + [p \cdot f \cdot] \; [g \cdot f] [r \cdot f \cdot] \; .$$

The first term is the incident electric field. The second term in this formula consists of three separate factors. These are called, in order, the phase factor, the geometric factor, and the reflectivity factor.

They arise as follows. The disturbance field $E(r) - E_0(r)$, derived from the thin shell approximation consists of a surface integral of the following type[1]

$$1.2) \qquad I = \int_S f(r)\, e^{ik\left\{|r-r_1| + |r-r_2|\right\}}\, ds$$

where r is a point (vector) on the surface, and r_1 and r_2 are points many wavelengths away. This is shown to be approximated by

$$1.3) \qquad I = f(\underline{r}) \int_s e^{ik\left\{|r-r_1| + |r-r_2|\right\}}\, ds$$

where the integration is over an appropriate small region s on S and the function may be evaluated at a convenient point $\underline{r}$ of the region. When this integral is transformed by a change of variables, it reduces to a product of Fresnel integrals over certain ranges. Now, the phase factor, $[p.f]$, in 1.1) includes these Fresnel integrals. The geometrical factor arises from the Jacobian of the above mentioned transformation, and the reflectivity factor constitutes the bulk of $f(\underline{r})$.

The function $f(\underline{r})$ depends upon a number of geometric parameters. The first section of this paper will therefore review quickly some of the elements of the differential geometry of surfaces. The next section will apply this to an explicit calculation of certain quantities needed in Keller's formulas. Then for the case that the surface of the obstacle is a quadric surface, another section will set up the equations from which one can find the so-called "specular" points, or points of stationary phase, when they exist. These "high-light" points, depending upon the total surface and the source and observation point, are the points whose neighborhoods contribute the preponderant part of the field reflected to the observation point.

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1. Keller, loc. cit. This integral is discussed in the Appendix to Keller's paper, pp. 38-43.

The formula[2] to be discussed is shown below. An entirely analogous formula is derived for $\bar{H}(r)$, but only the electric field will be discussed here. It is convenient to break the formula up into its three major parts. The definition of individual symbols of the formulas will be given in the sequel.

Of the three factors of $E(r)-E_o(r)$ indicated in 1.1), the first two are scalars and the third, the reflectivity factor, is a vector.

The first factor, called the phase factor, is the same for all components of the field $E(r)-E_o(r)$ and is given by[3]:

1.4p) $$[p\cdot f] = e^{iM'}\left\{F\left(L_A^{(2)}\right) - F\left(L_A^{(1)}\right)\right\}\left\{G\left(L_C^{(2)}\right) - G\left(L_C^{(1)}\right)\right\}$$

The second, the geometric factor, is given by

1.4q) $$[g\cdot f] = \frac{k_a \cos\alpha}{\pi D_2 J} .$$

The reflectivity factor may be written as

1.4r $$[r\cdot f] = \hat{n}[r\cdot f]_n + \hat{\tau}_1 [r\cdot f]_1 + \hat{\tau}_2 [r\cdot f]_2$$

where the components are as follows:

$$[r\cdot f]_n = \frac{k_a h}{2}\left[\mathcal{E}_n\left\{(1-\frac{\varepsilon'_a}{\varepsilon'_b})(\frac{2iG_m}{k_a} - \cos\alpha) + (\frac{\varepsilon'_b}{\varepsilon'_a} - 1)(i + \frac{2G_m}{k_a\cos\alpha})\left(i\cos\gamma + \frac{2G_m}{k_a}\left[1 - \frac{\varepsilon'_a}{\varepsilon'_b}\right]\right) + \right.\right.$$
$$\left.\left. + \left(\frac{k_b^2}{k_a^2} - 1\right)\frac{\varepsilon'_a}{\varepsilon'_b\cos\alpha}\right\}\right]$$

$$[r\cdot f]_1 = \frac{k_a h}{2}\left[\frac{i\omega}{k_a c}(\mu_b - \mu_a)\left(i\mathcal{H}_2\left[1 + \frac{\cos\gamma}{\cos\alpha}\right] + \frac{i\omega}{k_a c\cos\alpha}\left[\varepsilon'_b - \varepsilon'_a\right]\mathcal{E}_1 + i\frac{\cos\delta}{\cos\alpha}\mathcal{H}_n\left[\frac{\mu_a}{\mu_b} - 1\right]\right)\right.$$
$$\left. - (\frac{\varepsilon'_a}{\varepsilon'_b} - 1)\cos\beta\,\mathcal{E}_n + (1 - \frac{\varepsilon'_b}{\varepsilon'_a})\left(-\frac{\cos\gamma}{\cos\alpha} + \frac{2iG_m}{k_a\cos\alpha}\left[1 - \frac{\varepsilon'_a}{\varepsilon'_b}\right]\right)\mathcal{E}_n\cos\beta + (\frac{k_b^2}{k_a^2} - 1)\frac{\mathcal{E}_1}{\cos\alpha}\right]$$

$$[i\cdot f]_2 = \frac{k_a h}{2}\left[-\frac{i\omega}{k_a c}(\mu_b - \mu_a)\left(i\mathcal{H}_1\left[1 + \frac{\cos\gamma}{\cos\alpha}\right] - \frac{i\omega}{k_a c\cos\alpha}\left[\varepsilon'_b - \varepsilon'_a\right]\mathcal{E}_2 + \right.\right.$$
$$\left.\left. + i\frac{\cos\beta}{\cos\alpha}\mathcal{H}_n\left[\frac{\mu_a}{\mu_b} - 1\right]\right)\right.$$
$$\left. - (\frac{\varepsilon'_a}{\varepsilon'_b} - 1)\cos\delta\,\mathcal{E}_n + (1 - \frac{\varepsilon'_b}{\varepsilon'_a})\left(-\frac{\cos\gamma}{\cos\alpha} + \frac{2iG_m}{k_a\cos\alpha}\left[1 - \frac{\varepsilon'_a}{\varepsilon'_b}\right]\mathcal{E}_n\cos\delta\right.\right.$$
$$\left.\left. + (\frac{k_b^2}{k_a^2} - 1)\frac{\mathcal{E}_2}{\cos\alpha}\right)\right]$$

---

2. Keller, loc. cit. eq. 45), p. 25.

3. The letters F and G denote the evaluation of certain Fresnel integrals. These are discussed later.

In these formulas subscripts a and b refer to the homogeneous medium and the obstacle medium respectively. The ε' and μ are the complex dielectric and the magnetic parameters of the media and k, as usual, is defined by $k^2=\omega^2/c^2\varepsilon'\mu$, with c the velocity of light, and ω the frequency of the incident propagation. The script $\mathscr{E}_n$, $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{H}_n$, $\mathscr{H}_1$, $\mathscr{H}_2$ are the components of incident electric and magnetic field, and are assumed given. The letter h denotes the shell thickness, G_m is the mean curvature of the surface of the shell. The angles α, β, γ and other constants appearing in these formulas will be discussed in the sequel.

2. Differential Geometry of a Surface

We have the following situation: A surface reflects the light emitted from a point P_1 to a point P_2. We will call an arbitrary point of reflection on the surface: P_o

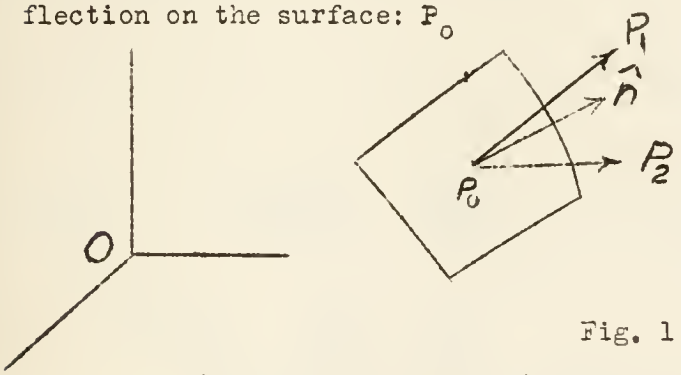

Fig. 1

Assuming an origin of coordinates, P_o, P_1, P_2 are at $\bar{r}_o, \bar{r}_1, \bar{r}_2$ respectively. Then the vectors of interest are:

2.1) $$\bar{r}_{o1} \equiv \bar{r}_1 - \bar{r}_o = |\bar{r}_1 - \bar{r}_o|\,\hat{r}_{o1} \equiv D_1\,\hat{r}_{o1}$$
$$\bar{r}_{o2} \equiv \bar{r}_2 - \bar{r}_o = |\bar{r}_2 - \bar{r}_o|\,\hat{r}_{o2} \equiv D_2\,\hat{r}_{o2}$$

(where a circumflex indicates a unit vector)

The unit normal vector to the surface at P_o lies along the gradient $\nabla\varphi$:

2.2) $$\hat{n} = \widehat{\nabla\varphi} = \frac{\nabla\varphi(\bar{r}_o)}{|\nabla\varphi|}$$

where $\varphi(x_o, y_o, z_o) = 0$ is the equation of the surface.

A convenient mode of specifying a surface is by means of three equations in two independent parameters:

2.3) $$x = x(u,v),\ y = y(u,v),\ z = z(u,v)$$

or briefly

$$\bar{r} = \bar{r}(u,v)$$

Now, $\frac{\partial \bar{r}}{\partial u}$ and $\frac{\partial \bar{r}}{\partial v}$ are tangent to the surface and determine a tangent plane.[4] We can form unit tangent vectors along the directions of variation of u and v respectively by the equations:

2.4) $$\hat{\tau}_1 = \frac{\partial \bar{r}}{\partial u} \Big/ \left|\frac{\partial \bar{r}}{\partial u}\right| = \frac{\tau_1}{|\tau_1|} \; ; \hat{\tau}_2 = \frac{\partial \bar{r}}{\partial v} \Big/ \left|\frac{\partial \bar{r}}{\partial v}\right| = \frac{\bar{\tau}_2}{|\bar{\tau}_2|}$$

and $\hat{n}$ (since it is perpendicular to τ_1 and τ_2) is also given by:

2.5) $$\hat{n} = \frac{\hat{\tau}_1 \times \hat{\tau}_2}{|\hat{\tau}_1 \times \bar{\tau}_2|}$$

The length element is given by:

2.6) $$ds^2 = dx^2 + dy^2 + dz^2 = (d\bar{r})^2 = \left(\sum_{\alpha} \frac{\partial \bar{r}}{\partial u^{\alpha}} du^{\alpha}\right)^2 \quad , \text{ or}$$

2.7) $$ds^2 = \sum_{\alpha,\beta} \left(\frac{\partial \bar{r}}{\partial u^{\alpha}} \cdot \frac{\partial \bar{r}}{\partial u\beta}\right) du^{\alpha} du\beta \equiv \sum_{\alpha,\beta} g\alpha\beta du^{\alpha} du\beta$$

where $g_{\alpha\beta} = \frac{\partial \bar{r}}{\partial u^{\alpha}} \cdot \frac{\partial \bar{r}}{\partial u\beta}$ [5]

The equation for any radius vector ending in the tangent plane is:

2.8) $$\bar{r}_p(u + du, v + dv) = \bar{r}(u,v) + \bar{\tau}_1 du + \bar{\tau}_2 dv$$

where du and dv are arbitrary. The formula for any radius vector ending in the surface $(\bar{r} = \bar{r}(u,v))$ and near enough to P_o is: (with $u^1 \equiv u$, $u^2 \equiv v$) .

---

4. Duschek - Mayer, Lehrbuch der Differentialgeometrie, Teubner, 1930, p. 85.

5. Blaschke, W. - Vorlesungen über Differentialgeometrie, Berlin, Springer, 1930.

2.9) $$\bar{r}\,(u + du,\ v + dv) = \bar{r}(u,v) + \sum_{\alpha=1}^{2} du^{\alpha}\,\frac{\partial \bar{r}}{\partial u^{\alpha}}$$

$$+ \frac{1}{2} \sum_{\alpha,\beta=1}^{2} du^{\alpha}\, du\beta\, \frac{\partial^{2}\bar{r}}{\partial u^{\alpha}\partial u\beta} + \ldots \text{ third order.}$$

Therefore, the vector from the tangent plane to the surface at (u+du,v+dv) is:

2.10) $$\Delta \bar{r} = \frac{1}{2} \sum_{\alpha,\beta} \frac{\partial^{2}\bar{r}}{\partial u^{\alpha}\partial u\beta}\, du^{\alpha}\, du\beta + \ldots$$

since $\bar{\tau}_{\alpha} = \dfrac{\partial \bar{r}}{\partial u^{\alpha}}$.

The projection of this on the normal is $\Delta\bar{r}\cdot\hat{n}$ $\left[=(\Delta r)_{n}\right]$. Therefore, denoting $(\Delta r)_{n}$ by p:

2.11) $$p = \frac{1}{2} \sum_{\alpha\beta} \hat{n}\cdot\frac{\partial^{2}\bar{r}}{\partial u^{\alpha}\partial u\beta}\, du^{\alpha}\, du\beta \equiv \frac{1}{2} \sum_{\alpha,\beta} d\alpha\beta\, du^{\alpha}\, du\beta$$

This formula is of great importance. It is called the second fundamental groundform[6]. The curvature of a second order curve element in the direction determined by given values of du and dv is given by:[7]

2.12) $$K = \frac{\sum_{\alpha\beta} d\alpha\beta\, du^{\alpha}\, du\beta}{\sum_{\alpha\beta} g\alpha\beta\, du^{\alpha}\, du\beta}$$

The principal directions are those determined by those sets (du,dv) which make K a maximum and a minimum. The conditions for this are:

2.13) $$\sum_{\beta} (d\alpha\beta - Kg\alpha\beta)\, du\beta = 0, \qquad \alpha = 1,2$$

and the condition that these be soluble is:

2.14) $$\begin{vmatrix} d_{11} - Kg_{11}, & d_{12} - Kg_{12} \\ d_{21} - Kg_{21}, & d_{22} - Kg_{22} \end{vmatrix} = 0$$

---

6. Blaschke, W. p. 54, Eq. 14.

7. ibid. p. 56, Eq. 22

There are two roots (K_1, K_2) to this equation in K. These are the principal curvatures. The solutions of (2.13) for $\frac{du^2}{du'}$ ($= \frac{dv}{du}$) give the directions of principal curvature[8]. The solutions of these equations ($\frac{dv}{du} = \psi_\alpha(u,v)$) give the lines of principal curvature in the form $C_\alpha = C_\alpha(u,v)$, $\alpha = 1,2$, and the C_α can be taken as possible "principal coordinates". These lines of principal curvature form two orthogonal families of surface curves.

These principal coordinates are associated with unit vectors $\hat{\tau}'_1$ and $\hat{\tau}'_2$, explicit formulas for which are given later in 3.14). Together with the unit normal $\hat{n}$, they give a new orthogonal coordinate system for space. This is the coordinate system in which the quantities to be calculated for Keller's formula are most directly expressed.

We observe that in a principal coordinate system, the length element is given by an expression of the form:[9] (since the lines of curvature are perpendicular).

2.15) $$ds^2 = g_{11}\, du^2 + g_{22}\, dv^2$$

If we now introduce the variables x_t and y_t so that

2.16) $$x_t = \sqrt{g_{11}}\ du \quad , \quad y_t = \sqrt{g_{22}}\ dv$$

then

2.17) $$ds^2 = x_t^2 + y_t^2$$

In these coordinates the formula for p will take the form[10]

2.18) $$p = \frac{1}{2}\left[\frac{d_{11}}{g_{11}}\, x_t^2 + \frac{d_{22}}{g_{22}}\, y_t^2\right] = \frac{1}{2} K_1\, x_t^2 + \frac{1}{2} K_2\, y_t^2$$

In the principal coordinate system, we can define the following quantities:

2.19)
$$\cos\alpha = \hat{r}_{o2} \cdot \hat{n} \qquad \cos\sigma = \hat{r}_{o1} \cdot \hat{\tau}'_2$$
$$\cos\beta = \hat{r}_{o1} \cdot \hat{\tau}'_1 \qquad \cos\varepsilon = \hat{r}_{o2} \cdot \hat{\tau}'_1$$
$$\cos\gamma = \hat{r}_{o1} \cdot \hat{n} \qquad \cos\zeta = \hat{r}_{o2} \cdot \hat{\tau}'_2$$

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8. D.M. p. 127.
9. D.M. p. 83, line 1
10. D.M., p. 130, Eqs. (11) and (14)

3. The Formula for the Scattered Field.

We now have enough differential geometry to evaluate specific quantities in Keller's formula. Recalling the definition of k_1 and of D_1 and D_2 in 2.1) and using 2.4), 2.5) and 2.19), it is possible to calculate the following quantities:[11]

3.1)
$$A = k_a \left\{ \frac{1}{2D_1} \left[1 - K_1 D_1 \cos\gamma - 1 \cos^2\beta\right] + \frac{1}{2} \frac{1}{D_2} \left[1 - K_1 D_2 \cos\alpha - 1 \cos^2\varepsilon\right] \right\}.$$

$$B = - k_a \left\{ \frac{\cos\beta \cos\delta}{D_1} - \frac{\cos\varepsilon \cos\zeta}{D_2} \right\}$$

$$C = k_a \left\{ \frac{1}{2D_1} \left[1 - K_2 D_1 \cos\gamma - 1 \cos^2\delta\right] + \frac{1}{2D_2} \left[1 - K_2 D_2 \cos\alpha - 1 \cos^2\zeta\right] \right\}$$

$$D = - k_a (\cos\beta + \cos\varepsilon)$$

$$E = - k_a (\cos\delta + \cos\zeta)$$

$$M = k_a (D_1 + D_2)$$

From these, we calculate ψ

3.2)
$$\tan 2\psi = \frac{B}{A-C}$$

and define:

3.3)
$$A' = A \cos^2\psi + B \cos\psi \sin\psi + C \sin^2\psi$$

$$C' = A \sin^2\psi - B \cos\psi \sin\psi + C \cos^2\psi$$

$$H = D \frac{\cos\psi + E \sin\psi}{2A'}$$

3.4)
$$K = E \frac{\cos\psi - D \sin\psi}{2C'}$$

$$M' = A'H^2 + C'K^2 - D'H - E'K + M$$

---

11. For convenience, Keller's quantities $\frac{x_1}{D_1}$ and $\frac{y_2}{D_2}$, etc. are replaced by the equivalent $\cos\beta$, $\cos\zeta$.

The functions F and G which appear in the expression for the phase-factor in 1.4p are integrals, functions of the upper limit of integration:

$$3.5) \qquad F(L_A) = \int_0^{L_A} e^{\frac{A'}{|A|}\frac{i\pi}{2}s^2}\, ds$$

$$G(L_C) = \int_0^{L_C} e^{\frac{C'}{|C|}\frac{i\pi}{2}s^2}\, ds \quad .$$

The real and imaginary parts of these integrals are the Fresnel integrals[12]. The integration may be thought of as carried out in the tangent plane to the surface, and the limits of integration correspond to that portion of the surface which may be assumed to contribute the major part of the reflection. This is discussed below. A discussion of the limits of integration is given in the Appendix I of Keller's paper. This discussion will be supplemented in the next paragraphs.

The original integrals (see 1.2)) from which Keller derives 3.5) have an integrand which oscillates on the surface of the obstacle and, according to the principle of stationary phase, the major contribution to the value of the integral is associated with the points at which the differential of the phase of the integrand vanishes. As Keller shows,[13] these points of stationary phase are the "highlight" points of the surface, the points of specular reflection. These, of course, are the points P_0 on the surface such that the normal at P_0 lies in a plane with the source and observation point and bisects the angle which these points subtend at P_0. The problem of finding these points when they exist is a simple one in principle and will be discussed later.

At the given point P_0 of the surface selected to be a point of stationary phase if such a point exists introduce coordinates by the rotation,

$$3.6) \qquad \tilde{x} = x_t \cos\psi + y_t \sin\psi + H$$
$$\tilde{y} = -x_t \sin\psi + y_t \cos\psi + K,$$

where ψ is defined as above, by 3.2).

Then the limits $L_A^{(\alpha)}$ and L_C^{α} may be expressed by

$$3.7) \qquad L_A^{(\alpha)} = \sqrt{\frac{2|A'|}{\pi}}\ \tilde{x}(\alpha)$$
$$L_C^{(\alpha)} = \sqrt{\frac{2|C'|}{\pi}}\ \tilde{y}(\alpha)$$

12. See Jahnke-Emde, Tables of Functions, Dover, N.J., Figure 15, p. 37, gives a plotting of these functions.
13. Appendix I, loc.cit. p. 39

Now if one looks at a plot[14] of the Fresnel integrals, one notes that they change with respect to the upper limit rather strongly near the origin. They begin to oscillate less strongly around a limiting value, when $x > 50$ or so, ($L_A > 6$). If, with Jahnke-Emde, we take the value of the integral from $-\infty$ to $+\infty$ to be 1, then the oscillations in the integral when $L_A > 6$ amount to less than .1 at the worst.

Thus, if $L_A^{(1)}$ and $L_A^{(2)}$ are both > 6 or both less than -6, the integral can have at most 1/10 of the value it has when $L_A^{(1)} < -6$ and $L_A^{(2)} > 6$, (i.e., when the region of integration includes the origin). Since the transformation (3.6) placed the origin at the stationary point, we see the justification for the assertion that the main contribution to the integral comes from the region surrounding a stationary point. Obviously, statements similar to the above apply to $L_C^{(1)}$ and $L_C^{(2)}$.

In order to obtain a rough criterion of the size of the region to be covered in order that the foregoing statements be accurate, we note that

3.8) $$\left| L_A^{(1)} \right| = \sqrt{\frac{2|A'|}{\pi}} \left| \tilde{x}_{(1)} \right| \sim \sqrt{\frac{2}{\pi} \cdot \frac{2\pi}{\lambda_a D}} \left| \tilde{x}_{(1)} \right| , \text{ where } D \sim \frac{1}{2}(D_1 + D_2),$$

and we wish $\left| L_A^{(1)} \right|$ to be > 6. Therefore,

$$\left| \tilde{x}_{(1)} \right| > \frac{6}{2} \sqrt{\lambda_a D} = 3\sqrt{\lambda_a D}$$

If $D = n\lambda_a$, then

$$\left| \tilde{x}_{(1)} \right| > 3\sqrt{n}\,\lambda_a \left(= \frac{3D}{\sqrt{n}}\right)$$

and taking n for example to be 100, $\left| \tilde{x}_{(1)} \right| > 30$. Therefore, if the piece of surface which is reflecting has edges which are > 30 wavelengths away from a specular point, the values of $\left[F(L_A^{(2)}) - F(L_A^{(1)})\right]$ and $\left[G(L_C^{(2)}) - G(L_C^{(1)})\right]$ will both be $< \frac{1}{10}$ the values they would have if the piece of surface covered a specular point. Therefore, their product, for the former case, will be $< \frac{1}{100}$ the product for the latter case. This indicates that, for n = 100 practically all the reflection emanates from a region about 20 to 30 wavelengths in radius surrounding a point of stationary phase.

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14. Janke-Emde, fig. 15: The variable x of these plots is equal to $\pi/2\, L_A^2$.

The quantities thus far calculated are needed for the phase-factor in Keller's formula, and constitute the bulk of the quantities needed for the other two factors. The geometric factor requires the calculation, also, of the simple quantity J,

3.9) $$J = 2/\pi \; |A'C'|^{1/2} = 2/\pi \; |AC - B^2/4|^{1/2} \quad .$$

This is to be substituted into 1.4g .

Finally, for the reflectivity factor 1.4r), one needs to know the form of the incident field. This is assumed by Keller to be of the general form:

3.10) $$\bar{E}_o = \bar{\mathcal{E}}(\bar{r})\, e^{ik_a|\bar{r}-\bar{r}_1|}$$
$$\bar{H}_o = \bar{\mathcal{H}}(\bar{r})\, e^{ik_a|\bar{r}-\bar{r}_1|}$$

where $\mathcal{E}$ and $\mathcal{H}$ are functions of position slowly varying with respect to the exponential. From these expressed in the original coordinate system, one has to calculate the components.

3.11) $$\mathcal{E}_n = \bar{\mathcal{E}}\cdot\hat{n}\,, \quad \mathcal{E}_1 = \mathcal{E}\cdot\hat{\tau}_1\,, \quad \mathcal{E}_2 = \mathcal{E}\cdot\hat{\tau}_2\,, \text{ and}$$
$$\mathcal{H}_n = \bar{\mathcal{H}}\cdot\hat{n}\,, \quad \mathcal{H}_1 = \mathcal{H}\cdot\hat{\tau}_1\,, \quad \mathcal{H}_2 = \mathcal{H}\cdot\hat{\tau}_2 \quad .$$

This calculation is easy, from formulas 2.4) and 2.5) and the given space dependence of $\mathcal{E}$ and $\mathcal{H}$.

To sum up the procedure for finding the first-order scattered field from Keller's formula:

We are given the position vectors $\bar{r}_o$, $\bar{r}_1$, $\bar{r}_2$; the initial field strength $\bar{\mathcal{E}}(\bar{r})$ and $\bar{\mathcal{H}}(\bar{r})$; and the shape of the surface at a point P_o in the form $\bar{r} = \bar{r}(u,v)$. We first calculate the relative displacement vectors given by Eq. 2.1). We then compute $\hat{\tau}_u$, $\hat{\tau}_v$, $\hat{n}$ from 2.4) and 2.2) or 2.5). We then calculate p and ds^2 from 2.11) and 2.7) . We exhibit p and ds^2 as sums of squares by solving 2.13) and 2.14) for K_1, K_2 $(\frac{dv}{du})_1$ and, $(\frac{dv}{du})_2$. Now let $(\frac{dv}{du_1}) \equiv \psi_1$ and $(\frac{dv}{du})_2 \equiv \psi_2$.

3.12) $$S_{(\alpha)} = g_{11} + 2g_{12}\psi_\alpha + g_{22}\psi_\alpha^2 \qquad \alpha = 1,2$$

Define:

3.13) $$\alpha_{\mu\nu} = \begin{bmatrix} \sqrt{\frac{g_{11}}{S_1}} & , \sqrt{\frac{g_{11}}{S_2}} \\ \sqrt{\frac{g_{22}}{S_1}}\,\psi_1 & , \sqrt{\frac{g_{22}}{S_2}}\,\psi_2 \end{bmatrix} \qquad \mu,\nu = 1,2$$

Then the unit tangent vectors in the direction of the lines of curvature ($\hat{\tau}'_\mu$) will be given in terms of those of the original parameterization ($\hat{\tau}_\nu$) by:

3.14) $$\hat{\tau}'_\mu = \sum_\nu \alpha_{\mu\nu} \hat{\tau}_\nu \qquad \mu,\nu = 1,2$$

Therefore, the transformation of coordinates from one frame to another is:

3.15) $$x'_\mu = \sum_\nu \alpha^{\mu\nu} \, x_\nu$$

where $$\alpha^{\mu\nu} = \frac{1}{\psi_2 - \psi_1} \begin{bmatrix} \sqrt{\frac{s_1}{g_{11}}}\,\psi_2 & , & -\sqrt{\frac{s_1}{g_{22}}} \\ \sqrt{\frac{s_2}{g_{11}}}\,\psi_1 & , & \sqrt{\frac{s_2}{g_{22}}} \end{bmatrix}$$

The equations 3.14) exhibit the principal vectors τ'_1 and τ'_2 on the surface in terms of the tangent vectors $\hat{\tau}_1$ and $\hat{\tau}_2$. The first pair, as we have seen, are intrinsically related to the geometry of the surface at the point P_o. The angles defined in 2.19) are defined in terms of these principal directions, and the normal n to the surface. They must be calculated with the use of 3.14).

Next, from 2.1), 2.18) and 2.19), we obtain the quantities necessary to calculate A,B,C,D,E and M in 3.1). From 3.3) and 3.4), we calculate A',C' and M'. We decide (in our $(\tilde{x},\tilde{y})$ coordinates given by 3.6) what our limits of integration are to be, calculate $L_A^{(\alpha)}$ and $L_C^{(\alpha)}$ in 3.7) and F and G of these arguments by 3.5). These are given, for example, by Jahnk-Emde.[15] The other necessary quantities are given by Eq. 3.11) and 3.12). In the next section we shall discuss the specialization of the previous paragraphs to the case that the reflecting surface is a central quadric.

4. Differential Geometry of Quadric Surfaces

In order to specialize the general procedures above to the case of a quadric surface, and in particular of an ellipsoid, it is advantageous to use ellipsoidal coordinates.

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15. Tables of Functions p. 35

These are defined by[16]

$$\frac{x^2}{a^2+w}+\frac{y^2}{b^2+w}+\frac{z^2}{c^2+w}=1 \qquad a>b>c>0,\quad w>-c^2$$

4.1)
$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1 \qquad -c^2>u>-b^2,\quad -b^2>v>-a^2$$

$$\frac{x^2}{a^2+v}+\frac{y^2}{b^2+v}+\frac{z^2}{c^2+v}=1$$

These variables u,v,w are roots of the same cubic equation. Each one of them is set equal to that one of the three roots of the cubic that falls within the limits indicated in Eq. 4.1). In order to deal with the surface of the ellipsoid we set w = 0 (This is the only variable which may be = 0). Solving for x,y,z, we have[17].

$$x(u,v) = \pm a\sqrt{\frac{(u+a^2)(v+a^2)}{-(a^2-b^2)(c^2-a^2)}}$$

4.2)
$$y(u,v) = \pm b\sqrt{\frac{(u+b^2)(v+b^2)}{-(b^2-c^2)(a^2-b^2)}}$$

$$z(u,v) = \pm c\sqrt{\frac{(u+c^2)(v+c^2)}{-(c^2-a^2)(b^2-c^2)}}$$

These equations can be solved for the parameters u and v with the result that:

$$u=\frac{1}{2}(x^2+y^2+z^2-a^2-b^2-c^2)+\frac{1}{2}\sqrt{\left[(a^2+b^2+c^2)+x^2+y^2+z^2\right]^2-4(a^2x^2+b^2y^2+c^2z^2+a^2b^2+b^2c^2+c^2a^2)}$$

and v is found from the same two terms with a minus sign between them.

---

16. Stratton, Electromagnetic Theory, p. 59, Eq. 133.

17. Stratton, loc. cit., with $\xi = 0$.

Thus, when we know the point P_o through its coordinates (x_o, y_o, z_o) we can find the corresponding u and v.

Using the matrix-notation to exhibit the components of a vector, we have the following results for the quantities of importance:

4.4) $$\frac{\partial \bar{r}}{\partial u} = \frac{1}{2} \begin{pmatrix} \sqrt{\dfrac{-a^2(v+a^2)}{(c^2-a^2)(a^2-b^2)(u+a^2)}} \\ \sqrt{\dfrac{-b^2(v+b^2)}{(a^2-b^2)(b^2-c^2)(u+b^2)}} \\ \sqrt{\dfrac{-c^2(v+c^2)}{(b^2-c^2)(c^2-a^2)(u+c^2)}} \end{pmatrix}$$, and

for $\frac{\partial \bar{r}}{\partial v}$, interchange u and v in the components of the vector above.

From these:

4.5) $$\hat{\tau}_u = \frac{1}{\sqrt{u(u-v)\,B}} \begin{pmatrix} \sqrt{-a^2(b^2-c^2)(v+a^2)(u+a^2)(u+c^2)} \\ \sqrt{-b^2(c^2-a^2)(v+b^2)(u+c^2)(u+a^2)} \\ \sqrt{-c^2(a^2-b^2)(v+c^2)(u+a^2)(u+b^2)} \end{pmatrix}$$

4.6) where $B = (a^2-b^2)(b^2-c^2)(c^2-a^2) < 0$

For $\hat{\tau}_v$, interchange u and v above.

From Eq. 2.5)

4.7) $$\hat{n} = \frac{-1}{\sqrt{-uv\,B}} \begin{pmatrix} \sqrt{b^2c^2(b^2-c^2)(u+a^2)(v+a^2)} \\ \sqrt{c^2a^2(c^2-a^2)(u+b^2)(v+b^2)} \\ \sqrt{a^2b^2(a^2-b^2)(u+c^2)(v+c^2)} \end{pmatrix}$$

From Eq. 2.7)[18], the matrix of the $g_{\alpha\beta}$, $\alpha, \beta = 1.2$ is

4.8) $$[g_{\alpha\beta}] = \frac{(u-v)}{4} \begin{bmatrix} \dfrac{u}{(u+a^2)(u+b^2)(u+c^2)} & 0 \\ 0 & \dfrac{-v}{(v+a^2)(v+b^2)(v+c^2)} \end{bmatrix}$$

---

[18]. Also see Stratton loc. cit.

Also:

4.9)
$$\frac{\partial^2 \bar{r}}{\partial u^2} = \frac{-1}{4\sqrt{B}} \begin{pmatrix} \left\{\sqrt{-a^2(b^2-c^2)(v+a^2)} \,/\, (u+a^2)^{3/2}\right\} \\ \left\{\sqrt{-b^2(c^2-a^2)(v+b^2)} \,/\, (u+b^2)^{3/2}\right\} \\ \left\{\sqrt{-c^2(a^2-b^2)(v+c^2)} \,/\, (u+c^2)^{3/2}\right\} \end{pmatrix} ;$$

for $\frac{\partial^2 \bar{r}}{\partial v^2}$ interchange u and v, above.

4.10)
$$\frac{\partial^2 \bar{r}}{\partial u \partial v} = \frac{1}{4\sqrt{B}} \begin{pmatrix} \left\{\sqrt{-a^2(b^2-c^2)} \,/\, \sqrt{(u+a^2)(v+a^2)}\right\} \\ \left\{\sqrt{-b^2(c^2-a^2)} \,/\, \sqrt{(u+b^2)(v+b^2)}\right\} \\ \left\{\sqrt{-c^2(a^2-b^2)} \,/\, \sqrt{(u+c^2)(v+c^2)}\right\} \end{pmatrix}$$

From 2.11) and 4.10), we obtain the matrix of the $d_{\alpha\beta}$, $\alpha, \beta = 1.2$:

4.11)
$$\left[d_{\alpha\beta}\right] = \frac{1}{4} \frac{abc}{\sqrt{uv}} (u-v) \begin{bmatrix} \frac{1}{(u+a^2)(u+b^2)(u+c^2)} & 0 \\ 0 & \frac{-1}{(v+a^2)(v+b^2)(v+c^2)} \end{bmatrix}$$

Thus $\left[d_{\alpha\beta}\right]$ is, in this case, a diagonal matrix. Solving 2.14), we obtain:

4.12)
$$K_{(1)} = + \frac{abc}{u\sqrt{uv}} , \quad K_{(2)} = + \frac{abc}{v\sqrt{uv}}$$

Therefore the quantities $\Psi_{(1)} \equiv (dv/du)_1$ and $\Psi_{(2)} \equiv (dv/du)_2$ obtained from the solutions of 2.13) are

4.13)
$$\Psi_{(1)} = 0 , \quad \Psi_{(2)} = \infty .$$

The quantities S_1 and S_2 of 3.12) are, in the present context,

4.14)
$$S_1 = g_{11} , \quad S_2 \sim g_{22} \Psi_{(2)}^2 ,$$

where g_{11} and g_{22} are as shown in 4.8) .

The coefficients $\alpha_{\mu\nu}$ of the transformation to principal vectors exhibited in 3.14 are now found as:

4.15) $$[\alpha_{\mu\nu}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It is of interest that, by replacing (u,v) by (v,w) and (w,u), we obtain results for hyperboloids of one and two sheets, respectively. Thus, the analysis is made a little more general by using ellipsoidal coordinates.

The formulas above are appropriate to the case where source and observation point are arbitrarily situated with respect to the quadric surface. They would be used in a numerical instance in which these points were specified by coordinates, and the surface given by numerical coefficients. However, if one considers the important case that source and observation point coincide, and if one further specializes the situation by supposing that this point lies on one of the principal axes of the ellipsoid then it is possible to derive some simple formulas which describe the scattered field. This will be carried out in the next section.

5. The Scattered Field

We suppose, now, that source and observation point coincide and lie on the z-axis of a general ellipsoid. The source will be taken as a source of dipole radiation, polarized along the x-axis. In calculating the phase factor 1.4f), we shall take the limits of the integrals F and G in 3.5) to be infinite.

The position vectors of the source and of the surface point P_o are given by

5.1) $$\bar{r}_1 = \hat{k}z_1, \quad \bar{r}_2 = \hat{k}z_1, \quad \bar{r}_o = \hat{k}\,c,$$

where $\hat{k}$ is the unit vector along the positive z-axis.

Then the vectors of 2.1) are:

5.2) $$\bar{r}_{o1} = \hat{k}(z - c), \quad \bar{r}_{o2} = \hat{k}(z - c)$$
$$\hat{r}_{o1} = \hat{k}, \quad \hat{r}_{o2} = \hat{k} \qquad (z > c)$$

From 4.3), $u = -b^2$, $v = -a^2$ and it follows that:

5.3) $$\hat{\tau}_2 = \hat{\tau}_u = \frac{1}{\sqrt{-b^2(a^2-b^2)}B} \begin{vmatrix} 0 \\ \sqrt{-b^2(c^2-a^2)(b^2-a^2)(c^2-b^2)(a^2-b^2)} \\ 0 \end{vmatrix} .$$

This means that $\hat{\tau}_u$ is the unit vector along the y-axis. Quite similarly, $\hat{\tau}_v$ is the unit vector along the x-axis. From these follows the obvious fact that $\hat{n}$ is the unit vector along the z-axis.

The directions $\hat{\tau}_u$ and $\hat{\tau}_v$ are easily seen to coincide with the principal directions $\hat{\tau}_u'$ and $\hat{\tau}_v'$, as exhibited in 3.14), and the cosines listed in 2.19) are determined by

5.4)
$$\cos\alpha = 1 \qquad \cos\delta = 0$$
$$\cos\beta = 0 \qquad \cos\epsilon = 0$$
$$\cos\gamma = 1 \qquad \cos\zeta = 0$$

From 4.12)

5.5) $$K_{(1)} = -\frac{c}{b^2}\ , \quad K_{(2)} = -\frac{c}{a^2}$$

Therefore, by 3.2)

5.6)
$$A = k_a\left\{\frac{1}{z-c}\left[\frac{1}{2} + \frac{c}{2a^2}(z-c)\right] + \frac{1}{z-c}\left[\frac{1}{2} + \frac{c}{2a^2}(z-c)\right]\right\}$$
$$= \frac{k_a}{z-c}\left[1 + \frac{c}{a^2}(z-c)\right] > 0 \quad \text{since } z >> c$$
$$C = \frac{k_a}{z-c}\left[1 + \frac{c}{b^2}(z-c)\right] > 0 \ .$$

$$B = 0\ , \quad D = 0\ , \quad E = 0\ , \quad M = 2k_a\ (z - ec)$$

$$\psi = 0, \quad A' = A, \quad C' = C, \quad M' = M$$

From 3.9) we get:

5.7) $$J = \frac{2}{\pi}\frac{k_a}{z-c}\sqrt{\left[1 + \frac{c}{b^2}(z-c)\right]\left[1 - \frac{ec}{a^2}(z-c)\right]}$$

Now, in order to have an explicit form for the reflection factor, we shall assume that the radiating dipole is of dipole strength $\vec{J}_o$ equal to $J_o\,\hat{j}$

The incident field in Keller's paper then becomes:

$$\bar{\mathcal{E}} = \frac{ik_a^2}{\omega\varepsilon'} \frac{J_o}{(z-ec)} \hat{i} \quad ; \quad \mathcal{E}_n = 0, \ \mathcal{E}_1 = \frac{ik_a^2 J_o}{\omega\varepsilon'(z-ec)} \ (=\mathcal{E}_o), \ \mathcal{E}_2 = 0$$

5.8)

$$\mathcal{H} = -\frac{ik_a J_o}{c(z-ec)} \hat{j} \quad ; \quad \bar{\mathcal{H}}_n = 0, \ \bar{\mathcal{H}}_1 = 0, \ \mathcal{H}_2 = -\frac{ik_a J_o}{c(z-ec)}$$

$$\mathcal{H}_2 = -\frac{\omega\varepsilon'}{k_a c} \mathcal{E}_1 = -\sqrt{\frac{\varepsilon'}{\mu}} \mathcal{E}_1 \ ,$$

where $\hat{i}$ and $\hat{j}$ denote unit vectors along the x and y-axes respectively.

We have chosen infinite limits for the integrals F and G. Now[19]

$$\int_{-\infty}^{\infty} e^{i\pi/2 \, s^2} ds = 1 + i \ ,$$

and $(1+i)^2 = 2i$. Making this substitution into the phase-factor 1.4f), and substituting the previously computed quantities into the other factors we can exhibit explicitly the electric radiation field at the observation point. This field is:

$$E = E_o + \left[\frac{e^{i2k_a(z-ec)}}{2} \quad [2i] \right] \left[\frac{1}{2\sqrt{\left[1 + \frac{c}{b^2}(z-c)\right]\left[1 + \frac{c}{a^2}(z-c)\right]}} \right]$$

5.9)

$$\left[\frac{k_a h}{2} \hat{j} \left\{ \frac{i\omega}{k_a c}(\mu_b - \mu_a)\left(-2i\sqrt{\frac{\varepsilon_a}{\mu_a}}\right) + \frac{i\omega}{k_a c}\left[\varepsilon'_b - \varepsilon'_a\right] + \left(\frac{k_b^2}{k_a^2} - 1\right) \right\} \mathcal{E}_o \right]$$

It is interesting that in this case of normal incidence, the curvature of the surface given by 5.5) does not enter into the "reflectivity factor" (third bracket). However, it does enter into the second factor of $E-E_o$; this is the so-called geometric factor.

We can easily calculate the geometrical factor for several well known cases by giving a,b,c different values.

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19. Jahnke-Emde, p. 35.

For a sphere (a=b=c)

$$[p\cdot f.] = \frac{1}{2}\left\{\left[1 + \frac{1}{c}(z-c)\right]\right\}^{-1/2} = \frac{1}{2}\left[1 + \frac{1}{c}(z-c)\right]^{-1}$$

For a cylinder ($a = \infty$, $b = c$)

$$[p\cdot f.] = \frac{1}{2}\left\{1 + \frac{1}{c}(z-c)\right\}^{-1/2}$$

For a plane $a = b = \infty$

$$[p\cdot f.] = \frac{1}{2}$$

These agree with Keller's results[20].

We will close this section with a single application of the formula 5.9) to the location of focal points. This problem is discussed in the Keller paper.[21] The existence of a focal point implies a situation in which the field strength of reflected radiation becomes infinite at some point of space. This will happen, in the case discussed above, if the second factor of 5.9) becomes infinite.

Inspection of this parenthesis shows that in that case either

$$1 + \frac{c}{b^2}(z-c) = 0 \quad \text{or} \quad 1 + \frac{c}{a^2}(z-c) = 0 .$$

This locates a focus at

$$s = c - \frac{b^2}{c} \quad \text{or} \quad z = c - \frac{a^2}{c} .$$

6. The Specular Points.

In order to make the most effective use of Keller's formula for a general surface and general arrangement of source and observation point one must locate the so-called specular points on the surface. These, as has been pointed out above, are the high-light points on the surface as seen from the observation point. These points may be located by the use of Fermat's principle that the path of reflected light from source to surface to observation point will be such as to minimize the length of the broken line path $SP_o + P_oO$, where S is the source point, P_o the specular point on the surface, and O the observation point.

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20. Keller. p. 29.
21. loc. cit. p. 28, also p. 42.

Thus if $\varphi(x,y,z) = 0$ is the equation of the surface, we must minimize

6.1) $$D = D_1 + D_2 = |\bar{r}_1 - \bar{r}_o| + |\bar{r}_2 - \bar{r}_o|$$

subject to the condition:

6.2) $$\varphi(x_o, y_o, z_o) = 0 .$$

The position vectors $\bar{r}_1$ and $\bar{r}_2$ are given, and it is the vector $\bar{r}_o$ that is to be found through its coordinates x_o, y_o, z_o. It is natural to do this by the Lagrange multiplier method, in which one seeks to minimize $D + \lambda\varphi$ with respect to the four independent parameters x_o, y_o, z_o, and λ [22].

Carrying this out in the case of the ellipsoid

$$\varphi(x,y,z) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

one is led to the following four equations from which x_o, y_o, and z_o can be determined.

6.3)
$$\frac{x_o - x_1}{D_1} + \frac{x_o - x_2}{D_2} = \frac{\lambda x_o}{a^2}$$

$$\frac{y_o - y_1}{D_1} + \frac{y_o - y_2}{D_2} = \frac{\lambda y_o}{b^2}$$

$$\frac{z_o - z_1}{D_1} + \frac{z_o - z_2}{D_2} = \frac{\lambda z_o}{c^2}$$

$$\frac{x_o^2}{a^2} + \frac{y_o^2}{b^2} + \frac{z_o^2}{c^2} = 1 .$$

This algebraic system gives rise to algebraic equations of the sixth degree, one in each of the unknowns x_o, y_o and z_o. These equations, in the general case, are somewhat too complex to be of interest here. The special case, possibly the most important one from point of view of applications, that the observation point is located at the source also leads to a sixth degree equation. This one, for the general ellipsoid, is as follows:

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22. See E.W. Wilson, Advanced Calculus, p. 120.

$$(b^2-a^2)^2\,(c^2-a^2)^2\;x_o^6+2a^2\,(b^2-a^2)(c^2-a^2)\left[b^2+c^2-2a^2\right]x_1x_o^5$$
$$+a^2\left\{a^2x_1^2\left[(b^2-a^2)^2+4(b^2-a^2)(c^2-a^2)+(c^2-a^2)^2\right]-(b^2-a^2)(c^2-a^2)\right\}x_o^4$$
$$+2a^4x_1\left\{a^2x_1^2\,(b^2+c^2-2a^2)+b^2y_1^2\,(c^2-a^2)+c^2z_1^2\,(b^2-a^2)\right.$$
$$\left.-(b^2-a^2)(c^2-a^2)\left[b^2+c^2-2a^2\right]\right\}x_o^3 \qquad (6.4)$$
$$+a^6x_1^2\left\{a^2x_1^2+b^2y_1^2+c^2z_1^2-\left[(b^2-a^2)^2+4(b^2-a^2)(c^2-a^2)+(c^2-a^2)^2\right]\right\}x_o^2$$
$$-2a^8x_1^3\,(b^2+c^2-2a^2)\,x_o-a^{10}x_1^4=0\,,$$

with a similar equation for y_o (and z_o).

In this equation x_1, y_1, z_1 are the known coordinates of the point which is both source and observation point.

It may be of interest to observe that even in the relatively simplest case that the reflecting surface is a sphere the finding of the specular points is not a trivial problem when source and observation point are arbitrarily disposed about the surface. Of course, in this case the problem treated above, that source and observation point coincide, does become trivial.

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N. Y. U. Institute of
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